mass in Hydrogen plus the mass in Helium then the Helium mass fraction is: (Mass of Helium)/(Mass of Hydrogen + Helium) or, in units of the proton mass, (4)/(4+12) = 1/4. The observed abundances of Deuterium, ^3He and ^4He are all quite consistent with simple predictions based on the theoretical conditions of the Universe at time 1–2 seconds (see Kernin and Sarkar 1996).

These three observations really form the observation foundation of the Hot Big Bang model. A fourth observation, that the Universe is filled with galaxies that are arranged in a complex structure, can not easily be accounted for in this model. While the general idea that structure formation via gravitational instability should produce observable anisotropies in the CMB is consistent with our observations, the overall complexity of the observed distribution of galaxies is not well understood yet. We will examine this issue in great detail in later chapters.

Overview of Relevant Cosmological Equations

The Robertson-Walker Metric

To place the Hot Big Bang model into a physical context necessitates a sensible mathematical formulation. To assist with this formulation we assume that the universe on a large scale is both homogeneous and isotropic. This assumption is known as the Cosmological Principle and the observed isotropy of the expansion and the CMB are strong testaments to its validity. If we accept this principle to be valid, then our task is to construct a geometrical model of the Universe that explicitly incorporates large scale homogeneity and isotropy. Ideally, this model should be described by a relatively small number of parameters, all of which can be observationally determined. Much of this book is devoted to a modern discussion of attempts to determine these parameters from observations. However, before doing that we must describe the framework that allows observations to be directly connected to our cosmological model.

To begin with, we note that General Relativity is a geometrical theory concerning the overall curvature of space-time. Within that context we seek to specify the coordinate properties of a homogeneous, isotropic, expanding Universe. If we are to fully describe the Universe in geometrical terms, we must derive a metric which describes the coordinate paths that objects are allowed to take. In deriving this metric we must introduce the concept of an **event**. An event is something which occurs at a certain place at a certain time. Hence all events in the universe can be thought of as occurring in a four-dimensional **spacetime** continuum, with three spatial dimensions and one dimension of time. To compute the separation between any two events in spacetime, it is necessary to specify the **metric**. As a simple example, consider the surface of a sphere, which can be thought of as a two-dimensional analogue to the four-dimensional spacetime. Using simple spherical trigonometry, the metric of a sphere can be written as

$$ds^2 = R^2 \left[(d\phi)^2 + cos^2\phi(d\theta)^2 \right] \tag{10}$$

where ds denotes the distance between two points on the surface of the sphere, R is the radius of the sphere, and $d\phi$ and $d\theta$ are the difference in latitude and longitude between the two points (measured in radians). With this expression, it is possible to compute the separation between any two points along the surface of the sphere. Hence the geometry of the sphere and the physical specification of events is completely described by its metric.

The geometry of four-dimensional spacetime is described by an analogous metric. However, instead of computing the distance between two points on the surface of a sphere, we wish to compute the separation between two events, which involves both space and time. Special relativity allows one to show that the spacetime interval, ds, between two events which occur near each other in flat space is given by

$$ds^2 = dt^2 - \frac{1}{c^2}\left(dx^2 + dy^2 + dz^2\right) \tag{11}$$

where dt is the time interval between the two events (as determined by an inertial observer), c is the speed of light, and dx, dy, dz correspond to the separation between the two events in each of the three spatial dimensions. Note that, unlike the metric for an ordinary sphere, the spacetime metric need not always be positive. The geometry of spacetime is completely specified by equation 11. A **geodesic** is the shortest interval between any two points in spacetime.

Equation 11 assumes a flat Euclidean geometry, in which initially parallel lines always remain parallel. However, according to Einstein's theory of General Relativity, spacetime is curved by gravity, which is a manifestation of the energy density of matter. The separation between two events will therefore depend on the curvature of spacetime. This is schematically shown in Figure 1-6 for three specific curvatures.

For a homogeneous and isotropic universe, the most general metric in curved spacetime is the **Robertson-Walker metric**, which was first proposed in 1934. Expressed in spherical polar coordinates (r, θ, ϕ), this metric takes the form

$$ds^2 = dt^2 - \frac{R^2(t)}{c^2}\left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 sin^2\theta d\phi^2\right] \tag{12}$$

where $R(t)$ is the **universal scale factor** which describes the time evolution of the universal expansion, k denotes the curvature of spacetime ($k = -1, 0, 1$ for negative, zero, or positive curvature), and the coordinate r is **comoving** with the universal expansion. If we imagine a particle at rest with a given set of coordinates r,θ, ϕ, then as long as no external forces operate on this particle. then the particle remains at those coordinates. These coordinates are said to be **comoving** coordinates. They are related to physical coordinates through the scale factor,

$$physicaldistance = R(t) * comovingdistance \tag{13}$$

The Robertson-walker metric is independent of any particular gravitational theory. Gravity enters through the scale factor $R(t)$ and the curvature constant k; the distance between any two spacetime events therefore depends on what specific cosmological model is adopted. This specific cosmological model is determined by the values of $R(t)$ and k which can be observationally determined.

The Dynamics of an Expanding Universe

Now that we have specified the metric that holds in a homogeneous and isotropic Universe, the next step is to consider the dynamics of an expanding Universe. Since the Universe has mass, then at all times the expansion must compete against the combined (attractive) gravitational acceleration of that matter. In Newtonian mechanics this acceleration is given by

$$\nabla^2\Phi = -4\pi G\rho \qquad (14)$$

where ρ is the matter density and Φ is the gravitational potential. This equation is historically known as Poisson's equation.

This simple equation for gravitational acceleration does not apply in the very early Universe due to the presence of very high energy photons. The majority of the mass-energy in the early Universe is in the form of radiation moving at c. The high radiation pressure drags the matter along with it and effectively counters the tendency for the matter to collapse. In this sense, the Universe acts as a relativistic fluid with a pressure term whose behavior is not adequately described by Newtonian mechanics. The details about the Stress-Energy tensor in Einstein's field equations are beyond the scope of this book (for reference see Weinberg 1972; Peebles 1993) but they lead to a generalization of equation 14:

$$\nabla^2\Phi = -4\pi G(\rho + \frac{3p}{c^2}) \qquad (15)$$

where $\frac{p}{c^2}$ is the pressure (which we subsequently set to p; c=1) and the combined term $\rho+3p$ effectively becomes the gravitational mass density ρ_g which produces the net gravitational acceleration of material that decreases the expansion rate.

If we now consider a sphere of radius r_s and volume V which has some mean gravitational mass density within it, the total mass of that sphere is given by

$$M = \rho_g * V = (4\pi/3)(\rho + 3p)r_s^3 \qquad (16)$$

The acceleration at the surface of the sphere is given by Newton's law of gravitation as $-GM/R_s^2$. Multiplying equation 13 by the term $-G/R_s^2$ then yields

$$acceleration = \ddot{r}_s = -(4\pi/3)G(\rho + 3p)r_s \qquad (17)$$

where $\ddot{r}_s^2$ refers to the second time derivative of the spatial coordinate, r_s (the first time derivative $\dot{r}_s$ is a velocity). Equation 17 is a standard equation in General Relativity and it describes the evolution (e.g., expansion or contraction) of a homogeneous and isotropic mass distribution. Within this sphere there is some net energy, E_n. This energy is ρV. If the material enclosed in r_s moves so that it changes r_s then E_n changes in accordance with how much work is done by the pressure of the fluid on the surface of the sphere. By conservation of energy we then have

$$dE_n = \rho dV + V d\rho = -p dV \tag{18}$$

Equation 18 states that the change in net energy is exactly equal to the change in volume multiplied by the pressure. Rearranging the terms involving dV and defining the volume as V = $\frac{4\pi}{3} r_s^3$ ($\dot{V} = 3\dot{r}_s r_s^2$)

$$\dot{\rho} = -(\rho + p)\frac{dV}{V} = -3(\rho + p)\frac{\dot{r}_s r_s^2}{r_s^3} = -3(\rho + p)\frac{\dot{r}_s}{r_s} \tag{19}$$

Solving for p in equation 19 and plugging that solution in equation 17 yields the following differential equation

$$\ddot{r}_s = (8\pi/3)G\rho r_s + (4\pi/3)G\dot{\rho}\frac{r_s^2}{\dot{r}_s} \tag{20}$$

This is a messy differential equation. If we multiply both sides by the term $\dot{r}_s$ and choose units such that the quantity $(4\pi/3)G$ =1 and let r_s be x then we arrive at the following functional form

$$\dot{x}\ddot{x} = 2\rho x\dot{x} + \dot{\rho}x^2 \tag{21}$$

Now the right hand side is just

$$\frac{d}{dt}(\rho x^2) \tag{22a}$$

and the left hand side is just

$$\frac{1}{2}\frac{d}{dt}(\dot{x}^2) \tag{22b}$$

Integrating both sides over time then yields

$$\frac{1}{2}\int \frac{d}{dt}(\dot{x}^2)dt = \int \frac{d}{dt}(\rho x^2)dt \tag{22c}$$

Switching back to normal units yields the first integral of equation 20:

$$\dot{r}_s^2 = (8\pi/3)G\rho r_s^2 + K \tag{23}$$

where K is a constant of integration which we can identify with the curvature term in the Robertson-Walker metric. Equations 17 and 23 are the main equations of this cosmological model.

If we consider the case of a static Universe where r_s is, by definition, constant and hence all derivatives are zero then equations 17 and 23 become

$$0 = 4\pi/3G(\rho + 3p); \quad 8\pi/3G\rho + K = 0 \tag{24}$$

Since the mass density ρ must be positive then to satisfy the constraint of a static Universe p must be negative. Since normal matter cannot have negative pressure, Einstein introduced the cosmological constant Λ into the field equations to serve as the source of negative pressure. In the static Universe Λ balances the net gravitational acceleration. But the Universe is not static, it is expanding according to the expansion scale factor R(t) given in equation 13. Our hypothetical sphere radius r_s will then be different at some later time, t, such that

$$r_s(t) = r_s(t = 0) * R(t) \tag{25}$$

Equation 17 now becomes