and the left hand side is just

$$\frac{1}{2}\frac{d}{dt}(\dot{x}^2) \tag{22b}$$

Integrating both sides over time then yields

$$\frac{1}{2}\int \frac{d}{dt}(\dot{x}^2)dt = \int \frac{d}{dt}(\rho x^2)dt \tag{22c}$$

Switching back to normal units yields the first integral of equation 20:

$$\dot{r}_s^2 = (8\pi/3)G\rho r_s^2 + K \tag{23}$$

where K is a constant of integration which we can identify with the curvature term in the Robertson-Walker metric. Equations 17 and 23 are the main equations of this cosmological model.

If we consider the case of a static Universe where r_s is, by definition, constant and hence all derivatives are zero then equations 17 and 23 become

$$0 = 4\pi/3G(\rho + 3p); \quad 8\pi/3G\rho + K = 0 \tag{24}$$

Since the mass density ρ must be positive then to satisfy the constraint of a static Universe p must be negative. Since normal matter cannot have negative pressure, Einstein introduced the cosmological constant Λ into the field equations to serve as the source of negative pressure. In the static Universe Λ balances the net gravitational acceleration. But the Universe is not static, it is expanding according to the expansion scale factor R(t) given in equation 13. Our hypothetical sphere radius r_s will then be different at some later time, t, such that

$$r_s(t) = r_s(t=0) * R(t) \tag{25}$$

Equation 17 now becomes

$$\frac{\ddot{R}(t)}{R} = -(4\pi/3)G(\rho + 3p) + \frac{\Lambda}{3} \tag{26}$$

where the first of the two terms on the right hand side is for the pressure and density of ordinary matter (e.g., stars and galaxies) and the second term includes the contribution of the Cosmological Constant. Equation 26 then describes the relativistic acceleration of the expansion, which in principle, can be dominated by Λ at late times as ρ and p decrease with time while Λ stays nearly constant. This in fact is the physical manifestation of non-zero Λ. In this case, the Universe evolves from being radiation dominated, to being matter dominated, to being vacuum energy dominated.

The second of the two cosmological equations, equation 23, can now be expressed as

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = (8\pi/3)G\rho + K/R(t)^2 + \frac{\Lambda}{3} \tag{27}$$

Equation 27 demonstrates that the rate of change of the scale factor R(t) is affected by three things: the net gravitational force acting to decelerate the Universe which is determined by the matter density (ρ); a curvature term related to the geometry of the Universe (K=0 is flat; K =+1 is positive curvature, K= -1 is negative curvature); and a term related to the vacuum energy which acts as a long range repulsive force.

The quantity $\frac{\dot{R}(t)}{R(t)}$ is the rate of change of the scale factor and is parameterized as H, the Hubble constant. H is a measurable quantity. For the case of $\Lambda = 0$ if we can measure H and ρ (the present day mass density of the Universe) then we will have solved for $K/R(t)^2$ and hence, under the Robertson-Walker metric, completely specify the geometry of spacetime. In this way, we have formulated a mechanism where observations can fully determine the cosmological model - there are no hidden variables. In the special case where the curvature of the Universe is zero (K=0) and $\Lambda = 0$, we have

$$H^2 = (8\pi/3)G\rho \tag{28}$$

The rate of change of ρ with time is given by equation 19 which can be rewritten in terms of the scale factor R(t) as

$$\dot{\rho} = -3(\rho + p)\frac{\dot{R}(t)}{R(t)} \tag{29}$$

With zero pressure (p=0) we have

$$\frac{\dot{\rho}}{\rho} = -3\frac{\dot{R}(t)}{R(t)} \tag{30}$$

The only solution to this is that $\rho \approx R(t)^{-3}$ ($\dot{\rho} = -3R(t)^{-4}\dot{R}(t)$). Note that the expression $\rho \approx R(t)^{-3}$ is also the one that satisfies the condition that the mass within a sphere $(4\pi/3\rho(t)R(t)^3)$ as a function of time remains constant. From equation 28 we then have

$$H^2 = \left(\frac{\dot{R}(t)}{R(t)}\right)^2 \sim R(t)^{-3}$$

which is only satisfied if the scale factor R(t) goes as $t^{2/3}$. The quantity $\frac{\dot{R}(t)}{R(t)}$ = H is now

$$H = \frac{2/3t^{-1/3}}{t^{2/3}} = \frac{2}{3t}$$

Thus, in the zero curvature case, the expansion age of the Universe is

$$T_{exp} = \frac{2}{3H} \tag{31}$$

If space is devoid of mass (and hence is negatively curved) then we can set ρ =0 to yield

$$\left(\frac{\dot{R}(t)}{R(t)}\right)^2 = \frac{K}{R(t)^2}$$

which has a solution of the form R(t) goes as t and the expansion age tends to H^{-1}. In either case, observations which determine H also then reveal the approximate age of the Universe. The fact that the ages of the oldest stars in the Universe are in the range .67 H^{-1} – 1.0 H^{-1} has led some (e.g., Liddle 1996) to claim that this forms another observational pillar for the Hot Big Bang theory. However, its important to emphasize that the expansion age of the Universe is only $\leq H^{-1}$ in the special case where $\Lambda = 0$. For

$\Lambda \geq 0$, the relationship with H^{-1} is considerably more complicated as Λ acts to make the expansion rate decrease more slowly as its a repulsive force. In fact, it can be shown that

$$H_o t_o = \frac{2}{3}\Lambda^{-1/2} ln\left(\frac{1+\Lambda^{1/2}}{(1-\Lambda)^{1/2)}}\right)$$

(see Kolb and Turner 1991; Krauss and Turner 1994) in the case of inflation (see chapter 4) in which $\Omega + \Lambda = 1$ (where $\Lambda = 0$ is the usual case). $H_o t_o$ ¿ 1 is satisfied for Λ ¿ 0.74. Hence, to reconcile the possible age problem by invoking non-zero Λ requires a value of Λ which is in significant excess of Ω meaning that at the present epoch the Universal is dominated by Vacuum Energy.

The final useful relation to derive is the expression for the critical density of the Universe. This is defined as the density which is required to eventually halt, by mutual gravitational contraction, the expansion of the Universe. Like the previous derivations, this one again can be done in terms of energy conservation assuming the Universe is a sphere of uniform density. In this case the total mass is given by equation 16 with $p = 0$. Consider now a galaxy trying to escape the surface of this sphere. Its potential energy is given by

$$PE = -GMm/R = \frac{-4\pi m r_s^2 \rho G}{3} \tag{32}$$

and its kinetic energy is given by

$$KE = 1/2mv^2; \quad (v = Hr_s) \tag{33}$$

The velocity, v, of this galaxy is determined by the expansion rate expressed by the Hubble law. Thus $v = Hr_s$. To escape this sphere of radius r_s, KE must exceed PE so we have the critical condition

$$1/2mH^2 r_s^2 + \frac{4\pi m r_s^2 \rho G}{3} \geq 0 \tag{34}$$

we can readily eliminate m and r_s^2 to arrive at the expression for critical density:

$$\rho_c = \frac{3H^2}{8\pi G} \tag{35}$$

The critical density is not dependent upon the size and mass of the Universe but only on its expansion rate. If the real density of Universe exceeds ρ_c then the Universe is destined to collapse. The Universe can not collapse at early times due to entropy production and the associated high radiation pressure. Equation 35 strictly only applies in the matter dominated era.

Modern Cosmological Puzzles

We close this first chapter by describing the topics we will pursue in detail throughout the rest of this book. . These topics are all relevant to the determination of H, ρ and Λ which are needed to fully specify our cosmology as previously discussed. Our approach will be to show how observations have been used to constrain these cosmological parameters. In addition to this topic, we will also focus attention on the dark matter content of the Universe and the nature of the dark matter as well as the formation of structure (galaxies and clusters) in the Universe. In general, galaxies and clusters of galaxies are used as test particles or probes to determine the cosmological parameters. However, most of these determinations are complicated by the unknown distribution and nature of the dark matter. The recognition of the importance of dark matter is the single biggest difference between current cosmological models and those that were popular a mere 20 years ago.

We devote a chapter to each of the following issues and use the most modern observations available to characterize our current state of knowledge:

1. What is the age of the universe as determined from the observed expansion rate and cosmological distance scale? Is there a need to invoke the cosmological constant to reconcile the ages of the oldest stars with the value of the Hubble constant?

2. What is the nature of the large-scale distribution of matter in the universe as traced by the three-dimensional galaxy distribution and do we have a sample that accurately characterizes it?