Boussinesq approximation (for ocean, atm. bound. layer, lab expts, ideal models) (6.6.4, but scaling not discussed)

Idea: Sometimes we can treat a compressible fluid as approximately incompressible. Formally, this is the Boussinesq approximation, which consists of:

(1) Neglecting density variations $\left(\frac{1}{\rho}\frac{D\rho}{Dt}\right)$ in the mass continuity equation to get

$$\nabla\cdot\vec{u} = 0 .$$

(filters out sound waves by effectively making $c_s = \infty$)

(2) Linearizing density variations in the momentum equation:

$$\frac{D\vec{u}}{Dt} = -\frac{1}{\rho_{00}}\nabla p' + B\hat{k} \quad , \quad B = -g\,\frac{\rho_\theta - \rho_{00}}{\rho_{00}}$$

p' is a pressure perturbation, ρ_{00} is a constant reference density.
ρ_θ is potl density.

Validity: We will show the Boussinesq approximation is valid if

(i) Density variations throughout the fluid are small (< 20%, say), so there is a reference density ρ_{00} such that $\frac{\rho - \rho_{00}}{\rho_{00}} \ll 1$ everywhere.

(ii) The generalized Mach number L/c_sT is small, where L is the lengthscale and T the timescale of the flow $(\le L/U)$

Unlike hydrostatic approx does not require $H \ll L$, so good for turbulence.

Derivation (Spiegel and Veronis 1960 Astrophysical J., 131, 442-447)

The derivation of (2) follows the initial steps of our derivation of the hydrostatic approx, but with a constant ρ_0 that may differ from the mean fluid density at any particular height z. This uses only assumption (i).

To derive (1), we must show $\left[\frac{1}{\rho}\frac{D\rho}{Dt}\right]$ is much less than the individual terms $\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}$ of $\nabla\cdot\vec{u}$, so the dominant balance in the mass continuity eqn. is of those individual terms with each other.

Let U be hor. velocity scale, L and H the horizontal/vertical lengthscales.

Then
$$\left[\frac{\partial u}{\partial x}, \frac{\partial v}{\partial y}, \frac{\partial w}{\partial z}\right] = \frac{U}{L} \quad , \quad [w] = U\frac{H}{L} \simeq W$$

Now, for an adiabatic flow on which entropy η is conserved following fluid parcels,

$$\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_\eta \frac{Dp}{Dt} = \frac{1}{\rho c_s^2}\frac{Dp}{Dt}$$

Partition $p = p_0(z) + p'(x,y,z,t)$ into hydrostatic part $p_0 = p_{00} - \rho_{00} g z$ and perturbation p'.

Then
$$\left[\frac{Dp_0}{Dt}\right] = \left[w\frac{\partial p_0}{\partial z}\right] = U\frac{H}{L}\rho_{00} g$$

$$\left[\frac{Dp'}{Dt}\right] = \frac{[p']}{T} = \frac{\rho_{00} U L}{T^2}$$

(using horizontal mom eqn $\frac{Du'}{Dt} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$ to scale p')