Thus

$$\left[\frac{1}{\rho c_s^2}\frac{Dp_0}{Dt}\right] = \left[\frac{1}{\rho c_s^2}\cdot w \cdot \underbrace{\frac{\partial p_0}{dz}}_{-\rho_0 g}\right] = \frac{gW}{c_s^2} = \frac{W}{H_\eta}$$ (H_η is adiabatic density scale height).

Since we have assumed small density variations throughout fluid (in particular in the vertical) we are obliged to restrict $\boxed{H \ll H_\eta}$

Hence

$$\left[\frac{1}{\rho c_s^2}\frac{Dp_0}{Dt}\right] << \left[\frac{\partial w}{\partial z}\right] = \frac{W}{H}$$

Also

$$\left[\frac{1}{\rho c_s^2}\frac{Dp'}{Dt}\right] = \frac{[p']}{\rho_{00}c_s^2 T} = \frac{\rho_{00} U L}{\rho_{00} c_s^2 T^2} = \frac{U}{L}\cdot\left(\frac{L}{c_s T}\right)^2$$

Since the generalized Mach number $L/c_s T \ll 1$,

$$\left[\frac{1}{\rho c_s^2}\frac{Dp'}{Dt}\right] << \left[\frac{\partial u}{\partial x}\right] = \frac{U}{L}$$

Consequently, both parts of $\frac{1}{\rho}\frac{D\rho}{Dt} = \frac{1}{\rho c_s^2}\frac{Dp}{Dt}$ are small compared to terms in the divergence, so the dominant balance is

$$\nabla\cdot\vec{u} = \nabla_h \cdot \vec{u}_h + \frac{\partial w}{\partial z} = 0$$ (incompressible continuity eqn).

<u>Proof of (2)</u>, we first note that since $\frac{\rho - \rho_{00}}{\rho_{00}} \ll 1$ throughout the fluid (small relative density variations),

$$-\frac{\nabla p'}{\rho} \approx -\frac{\nabla p'}{\rho_{00}}$$

Also we can simplify $b = -g\frac{\rho'}{\rho_0(z)}$, using potential density:

$$\frac{\rho_\theta'}{\rho_{00}} = \frac{\rho'}{\rho_0} - \frac{1}{\rho_0 c_s^2}p'$$

By scaling the vertical momentum equation, $[p'] = [\rho']gH$ for a density-driven flow, so

$$\frac{[p'/\rho_0 c_s^2]}{[\rho'/\rho_0]} = \frac{gH}{c_s^2} = \frac{H}{H_\eta} \ll 1$$

Using an adiabatic reference profile with potential density ρ_{00} ($\rho_{\theta 0} = \rho_{00}$) we conclude

$$b = -g\frac{\rho'}{\rho_0(z)} \approx -g\frac{\rho_\theta'}{\rho_{00}} = -g\left(\frac{\rho_\theta - \rho_{00}}{\rho_{00}}\right) = B$$, the Boussinesq buoyancy

Thus we've now demonstrated the second part of the Boussinesq approx.