Problem Set Number 3 Solutions:

1. $N_D = 0.13S_i^a$

For $N_D = 100$ all values of S_i are less than 100 so rain will never happen.

For $N_D = 10^7$ $S_i = 1.01$ for a = 3.93 so a must be greater than 3.93 for rain to occur.

2. Basically you just plug into:

$$v = \frac{2}{9}\frac{g\rho' r^2}{\eta}$$

to get 10^3, 10^5, 10^7 seconds but most of you did not answer the second part. For the large droplets the Reynolds number

$$R_e = \frac{\rho v r}{\eta}$$

Test for drag (i.e. < 1) computes to about 10 for the large droplet so you can not use the standard form for drag that goes into the first equation.

3

Solution: Putting $H = 0$ and $r_H = R$ into the equation before (6.30) we obtain

$$0 = 4\rho_l \int_{r_0}^{R} \frac{w - v_1}{v_1 E} \, dr_1$$

or, because E and w are assumed to be constant,

$$w \int_{r_0}^{R} \frac{dr_1}{v_1} = \int_{r_0}^{R} dr_1 = R - r_0$$

Therefore

$$R = r_0 + w \int_{r_0}^{R} \frac{dr_1}{v_1} \qquad (6.31)$$

Because $\int_{r_0}^{R} dr_1/v_1$ is a function only of R and r_0, it follows from (6.31) that R is a function only of r_0 and w. ■

4. Your numerical answer varies with assumption but anything between 50 and 150 years is pretty reasonable.

Use the equation for terminal velocity in problem 2 where density is the difference between the particle density (given) and the air density at the respective heights. But the air density is negligible. The particles are very small and these do have long residence times in the stratosphere.

I get a terminal velocity of 2.8 x 10^-6 m/sec (that is about 1 micron per second)

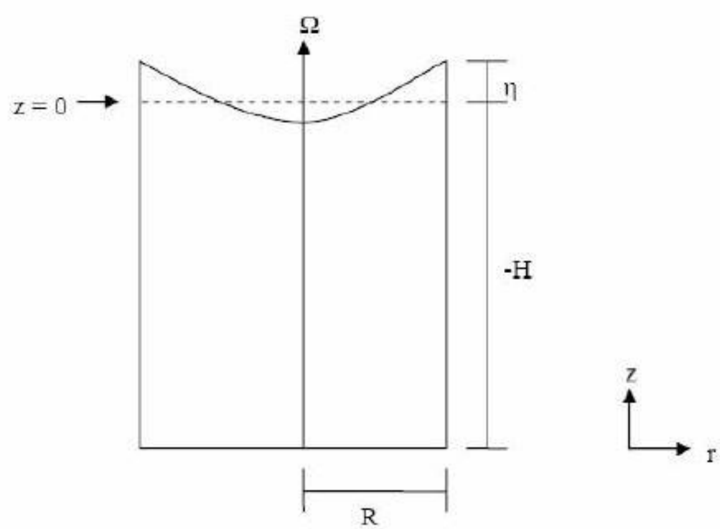
And a 200 mb height of 12000 km/s for 130 years.

Basically the ash particle slowly free falls through the stratosphere (since the background density is very small) and then frictionally drag through the troposphere ignoring all the winds.

Full Solution for Problem 5

1.a. Part One

The goal is to find the height that the free surface at the edge of a spinning beaker rises from its resting position. The first step of this process is to find an expression for the free surface height, η, as a function of radius, r, or distance from the center axis.



In this interpretation of the problem, the z axis is parallel to the rotational vector Ω. The radial axis is perpendicular to the z axis, with r = 0 at the center of the beaker. True gravity points downward along the z axis. Here, z = 0 is defined as the flat resting surface of the fluid. The height $\eta(r)$ is the distance between the fluid surface in a spinning beaker and z = 0. The depth of the fluid in a rest state is defined as z = -H. The radius of the beaker is R.

One way to approach this problem is by using a force balance. The momentum equation in a rotating frame of reference is given by

$$\frac{d\mathbf{u}}{dt} = -\frac{1}{\rho}\nabla p - \nabla\phi_{grav} + \Omega^2 r\mathbf{i} - 2\mathbf{\Omega}\times\mathbf{u},$$

where **i** denotes a unit vector along the r axis. In this case, since there is no velocity (in a rotating frame) and no acceleration, the momentum equation reduces to

$$0 = -\frac{1}{\rho}\nabla p - g\mathbf{k} + \Omega^2 r\mathbf{i},$$

In the vertical direction, there is simply hydrostatic balance:

$$0 = -\frac{1}{\rho}\frac{\partial\eta}{\partial z} - g.$$

Integrating this equation vertically from -H to η gives the pressure at the bottom,

$$p = p(-H) = \rho g(\eta + H) - p_{atm}.$$

In the horizontal, or radial, direction, the pressure gradient force balances the centrifugal acceleration:

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \Omega^2 r.$$

Substituting hydrostatic pressure p(-H),

$$0 = -g\frac{\partial\eta}{\partial r} + \Omega^2 r.$$

Integrating in the radial direction gives an (incomplete) expression for surface height as a function of radius:

$$\eta = \frac{\Omega^2 r^2}{2g} + c$$

An alternative approach is to treat the free surface as having constant geopotential:

Part Two

There are different methods to calculate the distance the free surface rises from its resting height, all involving volume conservation. The method presented here involves integration of circular shells.

Since mass (and therefore volume in a constant density fluid) is conserved, volume displaced upward from the resting surface z = 0 must be removed from below the resting surface.

$$\begin{aligned} 0 &= \int_0^R \left(\frac{\Omega^2 r^2}{2g} + c\right) 2\pi r dr \\ &= 2\pi \left(\frac{\Omega^2 R^4}{8g} + \frac{cR^2}{2}\right) \\ \rightarrow c &= \frac{\Omega^2 R^2}{4g} \end{aligned}$$

This means that difference between the resting surface and the height at the edge of the beaker is also $\frac{\Omega^2 R^2}{4g}$, one half of the total height difference between the center and edge.

1.b. In determining whether the bubbles will be closer together when they reach the surface, consider the bubbles at a time $t = 0$, just before they start to move. For a fluid parcel with no motion in a rotating reference frame, the full momentum equation is given by

$$0 = -\frac{1}{\rho}\nabla p - g\mathbf{k} + \Omega^2 r\mathbf{i}.$$

As seen in part a, the pressure gradients in the beaker are set up to balance true gravity and the centrifugal force. The bubbles will have much less density than the surrounding fluid, but the pressure exerted on them will be the same as any other fluid parcel located the same distance away from the center of the beaker. The first term in the above equation, the acceleration due to the pressure gradient force, will therefore increase in magnitude. Its direction stays the same.

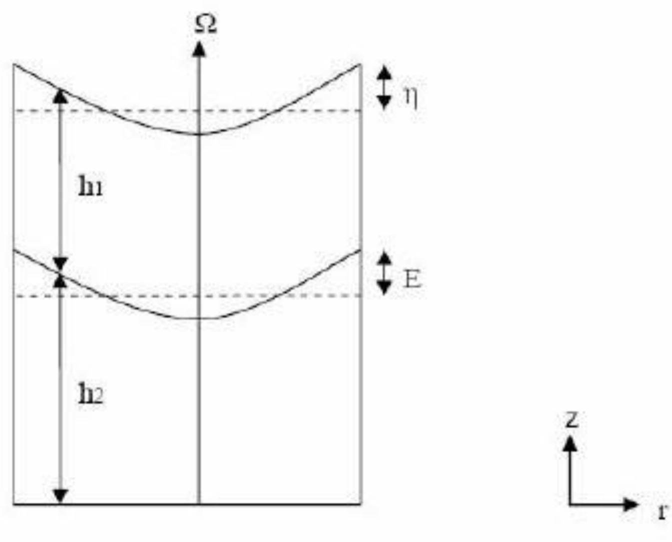
Motion will be opposite of the direction of the combined force of gravity and centrifugal force. In the center, there is no centrifugal force, so the motion of the bubble will be entirely vertical. Away from the center, the motion start out up and inward. The motion of the bubbles is perpendicular to the geopotential surfaces on which they are located.

Coriolis forces will only affect the bubble that is away from the center. Since the axis of rotation is vertical, the Coriolis force, $-2\mathbf{\Omega} \times \mathbf{u}$, will not affect the strictly vertical motion of the center bubble. This is just like the North Pole, where vertical velocity does not induce a Coriolis force because it is parallel to the axis of rotation.

The Coriolis force will deflect the horizontal motion of the outer bubble to the right. The magnitude of this force is proportional to the magnitude of the horizontal velocity of the bubble. So the bubble will feel no Coriolis force at time $t = 0$, then feel more Coriolis force as it accelerates inward and its horizontal velocity increases. The importance of Coriolis force will vary with the amount of rotation.

If the bubbles are small, they will experience strong frictional forces. This will keep the velocity of the bubbles slow, so Coriolis forces should only play a very minor role compared to bouyancy forces.

1.c. In a two layer fluid, it is possible to calculate the interface height in the same way as the free surface height in part a. Define E as the displacement of the interface from its resting height. The height between the interface and the surface is h_1 and the height between the bottom and the interface is h_2. The free surface height will have the same shape as in part a and will not be affected by the fluid layer below.



There is hydrostatic balance in the vertical direction. Integrating from the bottom to the surface gives the pressure at the bottom,

$$
\begin{aligned}
\int_{p}^{p_{atm}} \partial p &= -\int_{-H}^{\eta} \rho g \partial z \\
p &= \rho_1 g h_1 + \rho_2 g h_2 - p_{atm}
\end{aligned}
$$

At the bottom, since there is no velocity or acceleration in the rotating reference frame, the horizontal pressure gradient force balances the centrifugal force, as in part a.

$$
\begin{aligned}
\rho_2 \Omega^2 r &= \frac{\partial p}{\partial r} \\
\rho_2 \Omega^2 r &= \rho_1 g \left(\frac{\partial \eta}{\partial r} - \frac{\partial E}{\partial r} \right) + \rho_2 g \frac{\partial E}{\partial r} \\
\rho_2 \Omega^2 r &= \rho_1 g \frac{\partial \eta}{\partial r} - \Delta \rho g \frac{\partial E}{\partial r} \\
\rho_2 \Omega^2 r &= \rho_1 g \Omega^2 r - \Delta \rho g \frac{\partial E}{\partial r} \\
\Delta \rho g \frac{\partial E}{\partial r} &= \Delta \rho \Omega^2 r \\
\frac{\partial E}{\partial r} &= \frac{\Omega^2}{g} r
\end{aligned}
$$

Thus, the interface height is increasing away from the center. It will have the same shape as the free surface. This is consistent with the idea that geopotential surfaces depend only on gravity and centrifugal force, and therefore are not dependent on density.