Maximum Likelihood Method: MLM

Derived by Bernoulli in 1776 and again by Gauss around 1821; Worked out in detail By Fisher in 1812;

The MLE is a statistic that is
--efficient
--usually unbiased
--has minimum variance
--approx. Normally distributed

Consider the probability density function $F(x; \alpha)$, x a random variable, α a single parameter characterizing the known form of F. We want to estimate α. Let $x_1, x_2, \ldots, x_N$ be a random sample of size N, the x_i independent and drawn from the same population. Then the so-called ``likelihood function'' is the joint probability density function

$$\begin{aligned} L(x_1, x_2, \ldots, x_N) &= F(x_1, x_2, \ldots, x_N; \alpha) \\ &= F(x_1; \alpha) F(x_2; \alpha) \ldots F(x_N; \alpha) \\ &= \prod^{N} F(x_i; \alpha). \end{aligned}$$

This is the probability, given α, of obtaining the *observed* set of results. The maximum-likelihood estimator (MLE) of α is $\bar{\alpha}$ = (that value of α that maximizes $L(\alpha)$ for all variations of α), i.e.

$$\frac{\partial}{\partial \alpha} \ln L(\bar{\alpha}) = 0.$$

By way of example, Jauncey (1967) showed that ML was an excellent way of estimating the slope of the number - flux-density relation for extragalactic radio sources, and this particular application has made the technique familiar to astronomers. The source count is assumed to be of the power-law form

$$N(> S) = kS^{-\alpha},$$

where N is the number of sources on a particular patch of sky with flux densities greater than S, k is a constant and α is the exponent, or slope in the log N-log S plane, which we wish to estimate. If we consider M sources with flux densities S in the range S_0 to S_{max}, then a straightforward application of the ML procedure above yields the following likelihood function:

$$L(\alpha) = M \ln \alpha - \alpha \sum^{M} \ln s_i - M \ln(1 - b^{-\alpha}),$$

where

$$b = \frac{S_{max}}{S_0}$$

and

$$s_i = \frac{S_i}{S_0}.$$

Differentiation of this with respect to α then yields the equation from which $\overline{\alpha}$, the MLE of α, is obtained:

$$\frac{M}{\bar{\alpha}} - \sum^{N} \ln s_i - \frac{M \ln b}{b^{\bar{\alpha}} - 1} = 0.$$

Table 2. Candidate parameters: those which might be relevant for cosmological observations, but for which there is presently no convincing evidence requiring them. They are listed so as to take the value zero in the base cosmological model. Those above the gap are parameters of the background homogeneous cosmology, and those below describe the perturbations. Of the latter set, the first six refer to adiabatic perturbations, the next three to tensor perturbations, and the remainder to isocurvature perturbations.

Ω_k	spatial curvature
$N_\nu - 3.04$	effective number of neutrino species (CMBFAST definition)
m_{ν_i}	neutrino mass for species 'i'
	[or more complex neutrino properties]
$m_{\rm dm}$	(warm) dark matter mass
$w+1$	dark energy equation of state
$\mathrm{d}w/\mathrm{d}z$	redshift dependence of w
	[or more complex parametrization of dark energy evolution]
$c_S^2 - 1$	effects of dark energy sound speed
$1/r_{\rm top}$	topological identification scale
	[or more complex parametrization of non-trivial topology]
$\mathrm{d}\alpha/\mathrm{d}z$	redshift dependence of the fine structure constant
$\mathrm{d}G/\mathrm{d}z$	redshift dependence of the gravitational constant
$n-1$	scalar spectral index
$\mathrm{d}n/\mathrm{d}\ln k$	running of the scalar spectral index
$k_{\rm cut}$	large-scale cut-off in the spectrum
$A_{\rm feature}$	amplitude of spectral feature (peak, dip or step) ...
$k_{\rm feature}$	... and its scale
	[or adiabatic power spectrum amplitude parametrized in N bins]
$f_{\rm NL}$	quadratic contribution to primordial non-gaussianity
	[or more complex parametrization of non-gaussianity]
r	tensor-to-scalar ratio
$r + 8n_{\rm T}$	violation of the inflationary consistency equation
$\mathrm{d}n_{\rm T}/\mathrm{d}\ln k$	running of the tensor spectral index
P_S	CDM isocurvature perturbation ...
n_S	... and its spectral index ...
P_{SR}	... and its correlation with adiabatic perturbations ...
$n_{SR} - n_S$	... and the spectral index of that correlation
	[or more complicated multi-component isocurvature perturbation]
$G\mu$	cosmic string component of perturbations