PHYS 391 – Lab 2: Brownian Motion

Key Concepts

- Brownian Motion and Diffusion
- Video Analysis Techniques
- Statistical Description of Data
- Least-Squares Fitting

3.1 Introduction

The random motion of small particles when suspended in a liquid or gas has been recognized at least as long ago as 60 BC when the Roman Lucretius described this in the poem *On the Nature of Things* and used this as evidence for the existence of atoms. The name Brownian motion comes from the Scottish botanist Robert Brown, who made detailed observations of the motion of pollen in liquids under a microscope. In 1905, Einstein published a paper on Brownian motion (in addition to papers on special relativity and the photoelectric effect) providing a model for the phenomenon based on the kinetic theory of heat which tied together the Boltzmann constant, Avagadro's number, and gave some of the first hard evidence for the existence of atoms (in the modern sense of the word).

3.2 Goals of this Lab

This lab will observe Brownian motion using a digital USB microscope to observe the motion of micron-sized silicon spheres and, thus, testing the model of Brownian motion proposed by Einstein. By measuring the diffusion coefficient of the micro-spheres, a direct measurement of the Boltzmann constant will be inferred.

3.3 Theory

3.3.1 A Statistical Description of a Random Walk

Our model for Brownian motion is the constant collisions of silicon micro-spheres with water molecules. In one dimension, each collision can be approximated as an impulse which causes the particle to either move left or right with equal probability. For a series of n random steps, the probability of observing k steps to the left (and $n-k$ steps to the right) is equivalent to the probability of flipping heads on a coin k times out of n trials. This is also sometimes called a Random Walk. The probability of observing k out of n discrete events is described by the Binomial distribution

$$\frac{n!}{k!(n-k)!}p^k(1-p)^{n-k}$$

where p is the probability of a single discrete outcome and (1-p) is probability of the other discrete outcome. For an equally likely process ($p = 0.5$) like flipping a coin or a 1D random walk, this probability reduces to

$$\frac{n!}{k!(n-k)!}\frac{1}{2^n}$$

which in the limit of large n can be approximated very well by a Gaussian distribution with mean $\mu = n/2$ and $\sigma = \sqrt{n}/2$.

3.3.2 Relating Variance to the Diffusion Constant

If we want to think about the net displacement of our particle in one dimension $\Delta x = (x_2 - x_1)$ over some time interval $\Delta t = (t_2 - t_1)$, the model predicts this displacement will follow a Gaussian distribution with mean $\mu = 0$ and some characteristic width $\sigma_x^2 = 2D\Delta t$ where D is defined as the diffusion constant, which is simply a parameter which characterizes this motion for a given particle. The factor of 2 is included by convention. Even though in reality each collision imparts a varying amount of momentum to the object, the many collisions in any short time period can be viewed as an ensemble with an average value and variance, and the model of a random walk with a constant mean step size δ occurring every fixed time period τ still works well. Using the variance of the binomial distribution given above, one can then derive $D = \frac{\delta^2}{2\tau}$.

Observing Brownian motion, then, should lead to the following two results **which are to be verified in this lab.** The *average* displacement in any one dimension is $\overline{\Delta x} = 0$, which is to say on average the particle doesn't go anywhere, as it is equally probable to move left or right. The average squared displacement $\overline{(\Delta x)^2} = (\sigma_(Delta_x))^2 = 2D\Delta t$ (see [1]) however is non-zero, which is to say that the particle does on average travel some *absolute* distance from where it started, and that distance, given by the average RMS displacement, increases with time as $\sqrt{2D\Delta t}$. The motion of any specific particle can not be predicted, but if you were to look at the average motion of a series of particles, you should observe this characteristic statistical behavior. This notion of the statistical characteristics of a large ensemble of particles is central to the entire premise of thermodynamics and statistical mechanics.

For more dimensions, we should analyze the displacement in r rather than the 1D displacement in x. Since the motion in each dimension is uncorrelated, and $r^2 = x^2 + y^2 + z^2$, we immediately arrive at $\overline{(\Delta r)^2} = 6D\Delta t$ in 3 dimensions, and $\overline{(\Delta r)^2} = 4D\Delta t$ in 2 dimensions.

3.3.3 Relating the Diffusion Constant to Thermal Energy

The diffusion constant D depends on the size and shape of the diffusing particle, plus the nature of the medium it is suspended in. By thinking about drag forces, Einstein derived the relationship

$$Df = k_B T$$

where f if the drag coefficient relating the drag force to the velocity through the fluid ($F_{drag} = fv$) and $k_B T$ is the usual Boltzmann factor from statistical mechanics. One might think of this expression as relating the thermal kinetic energy of the particles to the energy lost to diffusion through the medium (likely through collisions with it). The Stokes formula gives the drag coefficient for a sphere in terms of the viscosity of the fluid and the radius of the sphere, R, as

$$f = 6\pi\eta R$$

where η is the viscosity of the fluid at temperature T. At room temperature, water has a nominal viscosity of $\eta = 1.0 \times 10^{-3}$ Pa s in SI units[2]. In practice, then, measuring the diffusion of a particle of known size from Brownian motion allows for a direct measurement of the Boltzmann constant k_B.

3.3.4 Measure Long-term Motion of a Single Particle (instead)

To observe and quantitatively characterize Brownian motion in the lab, we would really like to measure the average RMS motion of many different particles as a function of time. That is to say, we want to measure a large number of random walks and calculate the RMS displacement *on average* since any one random walk may deviate significantly from the average. In practice, this approach is rather tedious and involves a huge amount of time and labor collecting data. Since the net displacement of a single particle over two different time intervals is independent, an equivalent prescription is to track a single particle for a long time and compare the size of the individual displacements over each time slice Δt. The use of a computer-controlled camera to take pictures at regular time intervals makes this technique particularly convenient.

When you are taking real data, you should always strive to describe the conditions of data collection in your log book in as much detail as possible. Often, tracking systematic errors relies on having this sort of

[1] Remember, $(\sigma_{\Delta x})^2 = \overline{(\Delta x)^2} - (\overline{\Delta x})^2$, but in this case the mean displacement $\overline{\Delta x} = 0$ (but you should check this!). However, is the same true for $\overline{\Delta r}$?

[2] The units are Pascal seconds, or pressure multiplied by time

detail recorded. It is a good habit to try to be as descriptive as possible, so that you or somebody else can reproduce your results in the future if necessary. Including a rough sketch of your equipment is a good idea, as is recording the time when you do various things. Any raw data or other observations you make should be written (or taped) into your log book in a neat table. Any relevant uncertainties should also be noted at that time.

3.4 Observing Brownian Motion

Note: please check with your TA to ensure that these instructions are still accurate.

3.4.1 The OMAX Microscope

If you have spent a lot of time in Biology lab, these microscopes may be familiar. If you are like me, it may take a little playing around to figure out what you are doing.

Find a PC in room 17 with an available microscope and familiarize yourself with how it works. Obtain a sample slide from your TA and observe it under the microscope. This slide has deionized water with silicon microspheres in suspension. The *diameter* of these microspheres should be written down somewhere in room 17. There should be enough spheres that you can easily see several in the microscope frame, but not so many that they can't be distinguished. The slides are arranged so that the spheres mostly undergo 2D motion. Any significant motion in the vertical direction would cause the small beads to move out of the focal plane, and it would be exceedingly difficult to track the motion for any long period of time.

To find the microspheres, adjust the focus knob in the back and the translation knobs on the right-hand side so that you are looking at the water and have the focal plane in the water itself. It is very easy to focus on the top cover slip or bottom glass slide instead. If you are having trouble, first focus somewhere in the middle of the tape layer, then translate into the water drop in the middle. This is easiest done at low magnification, then you can switch to the x40 and a small adjustment of the focus should be all that is necessary (an 1/8th of a turn or less) to find some microspheres.

If you are really having trouble, get the calibration target and first figure out how to use the microscope on that instead. This slide has lines of known length painted on which we will use to determine the length scale of the video image we eventually record. The calibration target is much easier to find than the microspheres themselves.

3.4.2 Video Capture

Find something to look at in the microscope, then set up the video capture. This is most easily done within LoggerPro. Open a new LoggerPro window and select the *Insert:Video Capture* menu item. This will bring up a window showing the camera image on the microscope which is controlled over USB. Make sure you remember to turn on the microscope first, and make sure the light underneath is illuminating the sample with reasonable contrast. The camera has an automatic gain adjust, so varying the light won't actually have much effect on the image, although too much light will eventually saturate the camera and make it impossible to see anything.

You will want to take time-lapse movies with a period of at least $\Delta t = 5$ seconds with the microscope on its $\times 40$ power. Using a longer Δt (like 10 seconds) will make the motion of the beads more evident and easier to track, but you will need to wait longer to get the same number of points. The acquisition parameters are set by pressing the *Set Options* button. You should set up the acquisition to take a time-lapse movie with one frame every 5-10 seconds, and set this to run for a long enough time to get a large number of frames (at least 100). You may want to take a short trial run to test the entire process before you waste ten minutes collecting garbage data.

After setting the acquisition parameters, go back to the main window and click on *Start Time Lapse*. Observe the motion of several beads for a few minutes and convince yourself that you can both see Brownian motion, and also that there doesn't seem to be any net collective "drift" of the beads across the field of view. Try not to bump the desk, or do anything else that might cause additional motion in the video frame when the video is being recorded. The most likely reason for drift is evaporation from a bad seal, which will cause the water to drift towards the evaporation point. If you see this, try squeezing the cover slip onto the slide to try to seal the tape again, or else just try a different sample. Some small drift is acceptable, but

it shouldn't be obvious that everything is moving quickly in one direction. If the sample just came out of the refrigerator, give it some time to warm up. Another thing to watch out for is several beads which seem to be stuck together, as the interactions between the beads will dominate over the Brownian motion. Make sure you can observe at least one bead moving freely.

When the acquisition finishes, a movie window will be added to the main LoggerPro chart. You can close the camera window at that point, or just move it out of the way. Often the movie ends up exactly behind the camera window, so you may not even realize it is done.

When everything looks good, and you have a recorded video, you should be able to play it back and see the time-lapse motion of the beads. Make sure that things look as you would expect before you move on to the next step.

Ideally you should record the room temperature in your log book, although in the end the relative uncertainty on room temperature (in Kelvin) won't be significant even if you have to guess what it is.

3.5 Data Capture

To do any meaningful statistical analysis, we need to measure the position of a particle undergoing Brownian motion. To convert the movie from the microscope into position measurements, we will use the video analysis tool inside of LoggerPro.

In the bottom right-hand corner of the movie you just recorded, there should be an icon with red dots. Hover the cursor over this icon and the title "Enable video analysis" should pop up. Click on this icon to make other icons appear on the right edge of the video frame. Expanding the size of the video frame to fill the screen will probably make it easier to get good data, if you hadn't already done this.

Pick a bead which you will follow and record its position on every step. Click on the "Add Point" icon in the upper right and click on the bead you have chosen to follow. A single data point will be recorded into the spreadsheet recording the position of the bead within that frame. To get accurate data, you should try to choose some characteristic feature in the image of the bead to follow (such as a bright spot from the focusing of the backlight) and click on the same part of the image every time. The frame will advance automatically every time you select a point, so you can simply keep clicking to follow the bead from frame to frame.

Very soon you may have a big mass of selected points such that you can no longer see the bead. In the lower right there is an icon with a series of red points called "Toggle Trails". This will overlay the points as you take them which can toggle this feature on or off. I would turn it off until you get to the end, then you can turn it on to see the net motion across the entire movie. Keep adding points until you get through the entire movie, or at least have 100 points. If your chosen bead diffuses out of frame, just pick another one and follow it from that point on. The one anomalous step when you switch beads should be easy to spot in your final data sample and can be removed.

3.5.1 Data Export

To be able to read your data into MATLAB, you need to export the time and position data from LoggerPro into a text file. Either select the columns from the spreadsheet and directly copy/paste them into a text file, or use the File:Export As command. The latter will likely put extra stuff into the text file, but it should be straightforward to clean this up for importing into MATLAB. You only need the time and position data. If LoggerPro has also created a velocity column, you can safely delete this. You probably should save your LoggerPro session at this point as well, just in case you screw something up and need to come back to it. You can remove all of the data capture points by selecting *Clear All Data* in case you need to start over again.

3.5.2 Calibration Slide

In order to determine the length scale of the movie taken by the camera, a calibration image needs to be taken of some object of known length. There is a calibration slide available which has a tiny ruler on it which you can observe through the microscope. It may be difficult to find the target on the $\times 40$ power, so you may need to start at a lower magnification and get it well centered. Be very careful when switching magnification and focussing that you don't put the objective down on the calibration plate and crack it.

LoggerPro measures distances in the video images in units of pixels. Take a second short movie of the calibration target (a few frames is enough) so that you can convert pixels into a physically meaningful length. Either use the *Photo Distance* tool in the video analysis window to measure the length of the calibration target in pixels, or click a few points in the image of the calibration target (longer distances are more accurate). It is best to take data that will allow you to check the x and y calibration independently, so you should have at least two points separated by a large distance in x, and two other points separated by a large distance in y.

Sketch out in your log book what the target looks like, where the measurement was made, and the distances recorded both in pixels (from LoggerPro) and in actual length units (as specified on the calibration slide). Make sure you have an estimate of the uncertainty on the length scale of the calibration slide itself, plus possibly any uncertainty in your calibration procedure. Check whether your calibration (best is to report this in units of microns/pixel) appears to be the same in both directions. Make sure to note in detail what your are doing in your log book.

3.6 Data Analysis

Everything up to this point you could have done with a lab partner using LoggerPro and the microscopes. From here on, everything can be done in MATLAB, and each student should do their own work.

3.6.1 Data Calibration

The data you have collected is recorded as positions (in pixels) and time in the text file. Make sure the time value appears to match the actual clock time according to your acquisition parameters. If it does not, you will have to convert this to actual time by applying an appropriate scale factor.

The distance calibration step is most easily performed after the data has been read into MATLAB. Write down clearly what calibration parameters you are using for positions. Discuss how uncertain you think these length-scale calibrations might be, and explain whether you have used a common calibration parameter for x and y, or separate calibrations for each. Read the raw data from the text file into MATLAB, and apply the calibrations by simply multiplying the raw data by these parameters in your script.

3.6.2 Dispersion

We want to quantitatively measure the dispersion relation for Brownian motion. Read your raw data into MATLAB, calibrate the data as necessary, and write a script which turns the N position values in x and y into $N-1$ displacement values where each displacement is $\Delta x_i = x_i - x_{i-1}$. A handy trick to do this without looping over all values is to make two copies of the same data, remove the first element from one and the last element from the other, then subtract the two arrays. The first element of an array `x` can be removed with `x(1) = []`. Make displacement arrays for both x and y separately.

Make histograms of these arrays (Δx and Δy), and analyze this data to find the mean and standard deviation for each. For each, is the mean statistically consistent with zero? Is the standard deviation for Δx consistent with Δy? Use what we have learned in class to be as quantitative here as possible. In other words, how probable is it that the measured values in x and y correspond to the same fundamental value?

From your data, evaluate your best measured value for the dispersion constant D (including uncertainty). Explain in detail how you arrived at this result from your experimental data (describing the combination procedure and error propagation, for instance), and discuss which uncertainties dominate the final result. Remember that your calibration scale uncertainty may also effect the uncertainty on D.

3.6.3 Boltzmann Constant

Using what we know from Stokes Law (given in the introduction) convert your measurement of D into a measurement of the Boltzmann constant k_B. To estimate the uncertainty on k_B, explicitly write down each value which is needed to compute k_B (with uncertainty) and explicitly write out the error propagation formula for δk_B. In your error analysis, include any reasonable estimates of systematic uncertainties which may be important (or argue why they are not important). Quantitatively compare your measured value to

the accepted value of $k_B = 1.38 \times 10^{-23}$ J/K. You should quote how many standard deviations away your result is from the accepted value, and you should also indicate how probable that discrepancy is.

3.6.4 Time Evolution

We also want to empirically verify the time dependence predicted by the dispersion relation given above. In particular, we would like to see that the RMS displacement grows as $\sqrt{\Delta t}$. To avoid having to take more movies with different Δt intervals, instead we can analyze our data by making displacement measurements over longer time intervals. If instead of calculating $\Delta x_i = x_i - x_{i-1}$ we calculate $\Delta x_i = x_i - x_{i-2}$, we have effectively doubled the interval Δt.

Measure the RMS displacement as you did before, but now do this separately for several different Δt intervals (at least four). Put these values into a table and include the uncertainties on these points. Note that in principle these data for different Δt intervals are not strictly independent, but I suspect you won't notice the difference. You should be able to demonstrate a linear relationship between the variance of your displacement data and the time interval according to $\sigma^2 = 2D\Delta t$.

Make a scatter plot of your data and perform a linear fit to the data. How does the value of D found here (including an uncertainty) compare to the value of D that you measured in Section 3.6.2? Is there any evidence of deviations from the expected linear behavior? Make sure you are really plotting the correct things. As a cross-check, compare your data to a line constrained through the origin using the value of D that you measured in Section 3.6.2. You can just hand-draw this onto your plot if you wish. Does this line seem qualitatively consistent with your data?

3.7 Final Thoughts

Brownian motion was one of the first phenomena which gave direct evidence for the atomic nature of matter. By combining measurements of the Boltzmann constant in Brownian motion with PV measurements using the ideal gas law, physicists were able for the first time to get a direct estimate of Avagadro's number. Even knowing the order of magnitude of Avagadro's number was a major achievement at the time, and opened the door to many modern concepts which all derived from the understanding of the atomic nature of matter.