Phys 391 Lec 5

L3.1

1) normal dist. — recap on mean as best estimate

a) mean as best est.

~~b.) Sum of 2 normal vars~~

b) SDOM

2) a) weighted avg.

3) Brownian Motion Lab

1.) a.) mean as best

- Say we have N measurements from same exp. same parameter
  $x_1, x_2, \ldots x_N$

  G: need best estimate, $\bar{x}$, from these

- Say also all measurements follow a normal distribution ($G_{\bar{x},\sigma}(x)$)
  and if we knew $\bar{x}, \sigma$, we could calc.

  (via G) the probabilities of observing x_1, x_2 - etc.

(L3.2)

for example
from last time we said that

$$\text{Prob}([x_1, x_1+dx]) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x_1-\bar{x})^2}{2\sigma^2}} dx$$

- will drop the normalization term for now, and more formally discuss "joint probabilities"

$$\text{Prob}([x_2, x_2+dx]) \propto e^{\frac{-(x_2-\bar{x})^2}{2\sigma^2}}$$

etc.

So that probability of observing all N is

$$\text{Prob}_{\bar{x},\sigma}(x_1, x_2 \cdots x_N) = \text{Prob}(x_1) \cdot \text{Prob}(x_2) \cdots \cdot \text{Prob}(x_N)$$

→ multiplication ←

- now $e^y \cdot e^z = e^{(y+z)}$ ← sum exponents

$$\Rightarrow \text{Prob}_{\bar{x},\sigma}(x_1, x_2 \cdots x_N) \propto \frac{1}{\sigma^N} e^{-\sum[(x_i-\bar{x})^2/2\sigma^2]}$$

- we still don't know $\bar{x}$, σ however. Remember, our goal is to find them for this set x_i's

L3.3

To do this, we apply the "principle of maximum likelihood"
The best estimates $\bar{X}$ & σ are those that maximize $P(x_1, x_2 \ldots x_N)$

to maximize $P = \frac{1}{\sigma^N} e^{-\sum (x_i - \bar{X})^2 / 2\sigma^2}$ (5.40)

minimize the exponent

$$\sum_{i=1}^{N} \frac{(x_i - \bar{X})^2}{2\sigma^2}$$

take $\frac{\partial}{\partial \bar{X}}$ above $\Rightarrow$ $\sum_{i=1}^{N} (x_i - \bar{X}) = 0$

$\Rightarrow \bar{X} = \frac{\sum x_i}{N}$ □

$\Rightarrow$ So, given our N obs, $\bar{X} = \bar{x} = \frac{\sum x_i}{N}$ ∈
is our best estimate of $\bar{X}$

- to solve for best σ, set $\frac{\partial}{\partial \sigma}(5.40) = 0$ & solve. (HW problem 5.26)

$\Rightarrow$ $\sigma_b = \sqrt{\frac{1}{N} \sum (x_i - \bar{x})^2}$, in lieu of $\bar{X}$

Sometimes $\sigma_b = \sqrt{\frac{1}{N-1} \sum (x_i - \bar{x})^2}$

391 WS

(L3.9)

1 b.) SDOM

G: want to know the "reliability" of $\bar{x}$

- experimentally, we would consider taking N measures $(x_1, x_2 \ldots x_N)$ many (M) times, each time estimating

$$\bar{x}_k = \frac{\sum x_{i,k}}{N} \quad \forall k = 1, M$$

- rather, we would consider that each measure of x_i would also be normally dist, in the same manner. So would $x_2 \ldots x_N$

$$\Rightarrow \frac{\bar{x} + \bar{x} + \ldots \bar{x}}{N} = \bar{x}$$

- were interested in the "width" of the distribution

$$\sigma_{\bar{x}} = \sqrt{\left(\frac{\partial \bar{x}}{\partial x_1}\sigma_{x_1}\right)^2 + \ldots + \left(\frac{\partial \bar{x}}{\partial x_N}\sigma_{x_N}\right)^2}$$

(following general error prop. formula)

but $\sigma_{x_1} = \sigma_{x_2} = \ldots = \sigma_{x_N} = \sigma_x$

and $\frac{\partial \bar{x}}{\partial x_i} = \ldots = \frac{\partial \bar{x}}{\partial x_N} = \frac{1}{N}$

what we report as our SDOM

$$\Rightarrow \sigma_{\bar{x}} = \sqrt{\left(\frac{1}{N}\sigma_x\right)^2 + \ldots + \left(\frac{1}{N}\sigma_x\right)^2} = \frac{\sigma_x}{\sqrt{N}}$$

(L3,5)

2.) What if we now have the fabbers, 2 exp. measuring the same thing, different $\bar{x}$ + σ's. (assume both are norm. dists but with diff $\bar{x}$ + σ)

σ: How to combine?

We can again look at a joint probability + try to find $\bar{x}$ (x_{best}) that maximizes this prob.

$$JP = Prob_{\bar{x}}(\bar{x}_A + \bar{x}_B)$$

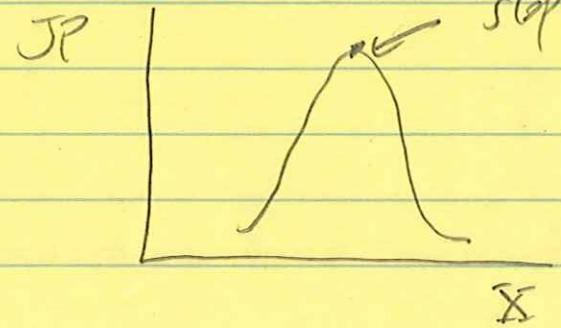
$$= \frac{1}{\sigma_A} e^{\frac{-(x_A-\bar{x})^2}{2\sigma_A^2}} \cdot \frac{1}{\sigma_B} e^{\frac{-(x_B-\bar{x})^2}{2\sigma_B^2}}$$

$$= \frac{1}{\sigma_A \sigma_B} e^{-\frac{\chi^2}{2}} \quad \text{"chi squared"}$$

with $\chi^2 \equiv \frac{(x_A-\bar{x})^2}{\sigma_A^2} + \frac{(x_B-\bar{x})^2}{\sigma_B^2}$

- Maximum likelihood entails:

$$\frac{\partial}{\partial \bar{x}}(JP) = 0$$



(L3.6)

entails minimizing χ^2

$$\Rightarrow \quad \chi^2 = \frac{(x_A - \bar{x})^2}{\sigma_A^2} + \frac{(x_B - \bar{x})^2}{\sigma_B^2}$$

solving $\Rightarrow$ $$\bar{x} = \frac{\frac{x_A}{\sigma_A^2} + \frac{x_B}{\sigma_B^2}}{\frac{1}{\sigma_A^2} + \frac{1}{\sigma_B^2}}$$

define $w_i = \frac{1}{\sigma_i^2}$ $\Rightarrow$

$$\bar{x} = \frac{w_A x_A + w_B x_B}{w_A + w_B}$$ weighted avg.

for more obs, turn into sum

$$\bar{x} = \frac{\sum w_i x_i}{\sum w_i} \qquad i = A, B, C \ldots \text{etc}$$

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3) B.M. Ch 6:

Phys 391 Exp

~~L3.8~~ WB L3.7

put ~5 μ silica spheres in deionized water @ room temp.

put drop on slide, look w microscope w/40x (or 100x) magnification. Make a movie

Q. What do you see?

Q Not much?, now set the "frame rate" of the movie to be 1 frame / 2-4 s

spheres appear to move

Q: why?, ok so why* $\sigma^2 = 2Dt$

Q: and what IS D?, or for that matter σ^2?

a diffusion constant, came from Einstein circa 1904-1906, very big year*

Q: why is D important?

b/c can use relationship $Df = k_B T$ to estimate k_B

so, guide, f can be calculated / viscosity of H2O

$f = 6\pi\eta R$ ← radius of sphere

* will explain later

~~L3.9~~ W15 L3.8

So, in lab, we'll make movies of our sphere undergoing "Brownian motion," estimate

1) make movies
2) estimate σ_x^2, σ_y^2, σ_r^2
3) 4 D's (3 of them)
4) use these to calc. k_B

But 6 NOTE, we use expression (for example 6)

$$\sigma_x^2 = \overline{\Delta x^2} - (\overline{\Delta x})^2 = 2D_x \Delta t$$

what is this? (is this zero?)

391 WS

L4.1

A.) L.S. fitting, "regression analysis"

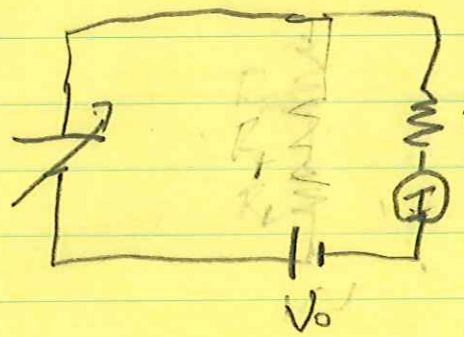
B) more on Brownian Motors Lab

A LS fitting

say we have N obs y_i in response to changing some independent variable x_i
eg. current I through circuit of resistance R while varying V across it

$I = \frac{V_0 - V}{R}$

$= \frac{V_0}{R} + \frac{V}{R}$



don't know this

eg.

will try to find "best fit line" through pts

$y = A + Bx$

$I = \frac{V_0}{R} + \frac{V}{R}$ $\quad B = \frac{1}{R}$

(L 4.2)

if I put subscript, i:

$$y_i = A + Bx_i$$

I implys there's an A, B for each pair (x_i, y_i)

instead, I assure there is some true y, Y_i for each x_i and note that $y_i \neq Y_i$ because of error distribution

The probability for a particular obsv, y_i, is: (assuming normal distribution)

$$P_Y(y_i) = \frac{1}{\sigma_y} e^{-\frac{(y_i - Y_i)^2}{2\sigma_y^2}}$$

$$= \frac{1}{\sigma_y} e^{-\frac{(y_i - (A+Bx_i))^2}{2\sigma_y^2}}$$

So the "trick" here is to assure that each obs y_i is to be compared to some "true" y, Y_i

asks for ⇒ So the probability of obs a series of y_i's associated w/ x_i's is (?)

$$JP \propto \frac{1}{(\sigma_y)^N} e^{-\frac{\chi^2}{2}}$$

with $\chi^2 \equiv \sum_{i=1}^{N} \frac{(y_i - A - Bx_i)^2}{\sigma^2}$ ← assuming σ const for all y_i's

L4.3

[note that were summing to get χ^2 b/c it refers to exp. of multiplied, exponential (Gaussian) probabilities]

we seek to maximize the J.P., ergo minimize χ^2

(Q) what do we do?

$$\frac{\partial}{\partial A}(JP) = 0 \quad + \quad \frac{\partial}{\partial B}(JP) = 0$$

what are we doing? finding A + B that maximize JP

$$\Rightarrow \quad AN + B\Sigma x_i = \Sigma y_i$$
$$+ \quad A\Sigma x_i + B\Sigma x_i^2 = \Sigma x_i y_i$$

solve 2 eqns for A, B. Could muck around algebraically (subst. etc) but that job calls for linear algebra!

write above as $\vec{M} \cdot \vec{F} = \vec{K}$ (matrix, vectors, vector)

with $\vec{M} = \begin{pmatrix} N & \Sigma x_i \\ \Sigma x_i & \Sigma x_i^2 \end{pmatrix}$, $\vec{F} = \begin{pmatrix} A \\ B \end{pmatrix}$

(all known)

$$+ \quad K = \begin{pmatrix} \Sigma y_i \\ \Sigma x_i y_i \end{pmatrix}$$ (vector)

L 4.4

we solve this how?

"divide" both sides by $\vec{M}$, weird in LE. this means

$$\vec{F} = INV(\vec{M}) \cdot \vec{K}$$

⇒ go to matlab polyol.m

what if variances (eg errors) are different for individual y_i's?

⟹ (imagine data from 2 or 3 experiments)

WRL for Wgts

individually weight data y_i's according to their individual errors

$$w_i \equiv \frac{1}{\sigma_i^2}$$

$$\Rightarrow \vec{M} = \begin{pmatrix} \Sigma w_i & \Sigma w_i x_i \\ \Sigma w_i x_i & \Sigma w_i x_i y_i \end{pmatrix}, \quad \vec{K} = \begin{pmatrix} \Sigma w_i y_i \\ \Sigma w_i x_i y_i \end{pmatrix}$$

show polyol_wgt.

L4.5

do polyfit_demo_script.m

what about higher order, how to "set up" eqns to solve?

eg. $y = A + Bx + Cx^2$

$$J.P. \propto e^{-\frac{\chi^2}{2}}$$

now $$\chi^2 \equiv \sum_{i=1}^{N} \frac{(y_i - A - Bx_i - Cx_i^2)^2}{\sigma_i^2}$$

see how it works $(y_i - \text{fit fctn})^2$ (in numerator)

B) more on Brown Motion

PHYS 395
W 19
Lec 6

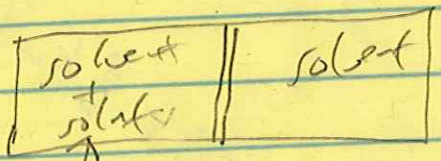
WIS
1

L4.6

1) a bit about Brownian Motion
   a) Osmosis as a model
   b) Langevin's contribution
2) this week's lab
3) Matlab sketches for WRS ??
4) student presentations

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1 a) consider a system of solute + solvent, former large molecule.



all at same pressure

dilute

barrier permeable to solvent only

what happens next?

* solute contributions to P from left side
* fewer solvent mol crossing L→R than R→L
* not solvent molecules flow to L
* equil in flow reduced

Pkets 3rd
W 9
Le 6

W15

(14.7)

Einstein (1905)
* P gets higher on LHS, membrane moves right
* solute molecules can be viewed as many "ideal gas molecules in vacuum"
* Brownian particles analogous to solute molecules

$$\omega = \overline{KE} = \frac{3}{2} k_B T$$

---

b Langevin

"follow the Brownian particle"

$$m\left(\frac{dx}{dt}\right)^2 = k_B T$$

eq. of motion $m\frac{d^2x}{dt^2} = -6\pi a\eta \frac{dx}{dt} + X$

(viscous drag) ↓ ; drag from Stokes law ↙ ; ↑ net force by "solvent"

Manipulations $x m\frac{d^2x}{dt^2} = -6\pi a\eta x\frac{dx}{dt} + xX$

Clever rewrite & average over $m\frac{d}{dt}\left(x\frac{dx}{dt}\right) - m\left(\frac{dx}{dt}\right)^2 = -3\pi a\eta\frac{d}{dt}x^2 + Xx$

Now average over "long time" ← relates to little collision nature of effect on big

$$\overline{m\frac{d}{dt}\left(x\frac{dx}{dt}\right)} - \underbrace{\overline{m\left(\frac{dx}{dt}\right)^2}}_{k_b T} = \overline{-3\pi a\eta\frac{d}{dt}x^2} + \cancel{\overline{Xx}}^{\,0}$$

Phys 344 ~~Q4~~ ~~Lec G~~ WI5

~~3~~ Lec B

can commute taking derivative w/ average:

$$\Rightarrow \frac{m}{2}\frac{d^2}{dt^2}\overline{x^2} + 3\pi a\eta \frac{d}{dt}\overline{x^2} = k_B T$$

need to solve, set $y = \frac{d}{dt}\overline{x^2}$

$$\Rightarrow \frac{dy}{dt} + \frac{6\pi a\eta}{m} y = \frac{2k_B T}{m}$$

$$\Downarrow$$

soln $$y(t) = \frac{k_B T}{3\pi a\eta} + Ce^{-\frac{6\pi a\eta}{m}t}$$

2nd term on right "dies off in microseconds"

$$\text{so } y(t) = \frac{d}{dt}\overline{x^2} = \frac{k_B T}{3\pi a\eta}$$

$$\Rightarrow \overline{x(t)^2} = \frac{k_B T}{3\pi a\eta} t$$

$$\text{or } \overline{\Delta x^2} = \sigma_x^2 = 2D\Delta t$$

$$\Rightarrow 2D = \frac{k_B T}{6\pi a\eta}$$

in lab: $\overline{(\Delta x)^2} = \sigma_x^2 = 2D\Delta t$

and $$Df = k_B T$$

$$\text{So } f = 6\pi a\eta$$