Please see WKPWS for text

2) Covariance (A)

assume we're interested in some function $q(x,y)$ that "governs" some physics relationship

[ex: $a_g = \frac{2h}{t^2}$]

so we do N trials, trying to keep x & y the same each time; and we estimate N q_i's

$$q_i = q(x_i, y_i)$$

we assume uncertainties are reasonably small and thus assume:

$$q_i \approx q(\bar{x}, \bar{y}) + \frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})$$

(from the "real" q taken @ $(\bar{x}, \bar{y})$)

We're interested in $\bar{q}$ which should be the "best estimate" for the "real" q:

$$\bar{q} = \frac{1}{N}\sum_{i=1}^{N} q_i = \frac{1}{N}\sum\left[q(\bar{x},\bar{y}) + \frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})\right]$$

2nd, 3rd terms give zero (why?)

so

$$\bar{q} = q(\bar{x}, \bar{y}) \quad \left[\text{e.g. } \bar{g} = \frac{2\bar{h}}{(\bar{t})^2}\right]$$

Please see this link for 4a6

~~(1)~~ (2)

what's more interesting is σ_q, with

$$\sigma_q^2 = \frac{1}{N}\sum (q_i - \bar{q})^2$$

? how could we calculate

Subst. [1] + [2] $\Rightarrow$

$$\sigma_q^2 = \frac{1}{N}\sum_i \left[\frac{\partial q}{\partial x}(x_i - \bar{x}) + \frac{\partial q}{\partial y}(y_i - \bar{y})\right]^2$$

$\Rightarrow$ this the squared term that leads to covariance

$$= \left(\frac{\partial q}{\partial x}\right)^2 \frac{1}{N}\sum (x_i - \bar{x})^2 + \left(\frac{\partial q}{\partial y}\right)^2 \frac{1}{N}\sum (y_i - \bar{y})^2$$

(σ_x^2 ... σ_y^2)

$$+ 2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y}\frac{1}{N}\sum (x_i - \bar{x})(y_i - \bar{y})$$

$\Rightarrow$ $\sigma_q^2 = \left(\frac{\partial q}{\partial x}\right)^2 \sigma_x^2 + \left(\frac{\partial q}{\partial y}\right)^2 + \sigma_y^2 + 2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y}\sigma_{xy}$

where $\sigma_{xy} \equiv \frac{1}{N}\sum (x_i - \bar{x})\cdot(y_i - \bar{y})$

$\Rightarrow$ this works regardless if measures of x + y are independent or not

$\Rightarrow$ indeed $2\frac{\partial q}{\partial x}\frac{\partial q}{\partial y}\sigma_{xy}$ is a measure of "independence of" (or conversely" correlation of") x + y

Scenario where $x_i > \bar{x} \Rightarrow y_i > \bar{y}$ consistently

(Plus 3pt with 15 [illegible])

(4)
(3)

Covariance (B) & Goodness of fit

For a different set of pairs (x_i, y_i) where they aren't trying to be kept the same, we can postulate that they obey a linear relationship

$$y = A + Bx$$

(as we did with least-squares fitting)

we assert that, if x_i, y_i's indeed are thusly correlated

$$r \equiv \frac{\sigma_{xy}}{\sigma_x \sigma_y}$$

(linear correlation coeff)

will be a measure of "goodness of fit"

[remember $r \equiv \frac{\sum (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}$

just comes out of the numbers]

Note that r will take values $-1 \leq r \leq 1$

- If every (x_i, y_i) fall perfectly on line

then $(y_i - \bar{y}) = B(x_i - \bar{x})$ for all i's

So,

[Pg. 75 3711 Wells Lec #6]

$$r = \frac{B\sum(x_i-\bar{x})^2}{\sqrt{\sum(x_i-\bar{x})^2 B^2 \sum(x_i-\bar{x})^2}} = \frac{B}{|B|} = \pm 1$$

for no fit, top of $r \to 0$

3) Poisson

(background)

a scenario where we have a probability of decay (p) and its error (δp)

assume p (prob of decay of a nucleus) is small and large sample gives large N, # of "trials" in time Δt

eg. $N \sim 10^{25}$ atoms

$\Rightarrow$ how many decays, ν, would we expect in time Δt

use Poisson: $P_\mu(\nu) = e^{-\mu}\frac{\mu^\nu}{\nu!}$

$$\mu = pN \qquad (p \ll 1)$$

so μ ($\mu > 0$) is expected # of mean counts

and ν ($\nu = 0$) is # decays in time Δt

$P_\mu(\nu)$ gives probability for ν decays given expected μ

Plan 391 with Lecture 6 ⑥ ⑤

Lecture 7

18 hours each layer 1 egg/day
check between hours, ~odds of finding
0, 1, 2, ... etc eggs?

rate $\frac{18 \text{ eggs}}{24 \text{ hours}} = 0.75 = \mu$

$\left(= \frac{1}{24\text{hr}} \cdot {}^{18}\text{eggs} = p \cdot N\right)$

what are r's ? w.r.t

0; 1; 2; 3; etc
up to ?? (18?) but Poisson

do MC example (lecture7.m part 3)

Notes: * parts 1 + 2 give more gaussian behavior as $\mu \rightarrow$ larger

* Lecture 7 does histogram of some counting data

* later sections give "brute force" fitting method & results

Plot 3pt ~~Sep~~ W/S ~~Left~~ →

~~4~~ 6

3) Fourier

$$\Psi(\omega) \int_{-\infty}^{\infty} \Psi(t) e^{-i\omega t} dt$$

(done?)

ok define IFT $\Psi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Psi(\omega) e^{+i\omega t} d\omega$

Questions?

* have to know $\Psi(t)$ (or $\Psi(\omega)$) for all time, continuously

(So now lets rejoin the real world ...)

real data is discrete (meaning?)

on board first, then ML

ML → Lecture_8_.ekz (AS + AA)

(mention sample freq ν_s)

Rule! can't "see" freqs of $> \frac{1}{2}\nu_s$

Called the Nyquist $\nu_N = \frac{1}{2}\nu_s$

Plus Bar
Sup by Wolfs
Lec Topics

(8)

(7)

consider a "time series" of N points

He $T_{sample} = N\Delta t = \frac{N}{\nu_s}$

Can't observe frequencies $< \frac{1}{T_{sample}} = \frac{\nu_s}{N}$

thus sets a range of frequencies we can observe

Given ν_s & N

Dr. Ds rules of thumb

$\nu_c < \nu < \nu_n$

$\frac{3\nu_s}{N}$ $\qquad$ $\frac{\nu_s}{4}$

Ok, are we done? No, there's aliasing & foldover

MC ⟹ Lecture 8... 1 B Hem C Part of them

---

aliasing means that signals above the sample rate "fold-over" and look like legitimate data.

~~Day~~ W6

~~L7.2~~

L7.1

2. Fourier

$$\Psi(\omega) = \int_{-\infty}^{\infty} \psi(t)\, e^{-i\omega t}\, dt$$

and IFT: $\psi(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \Psi(\omega)\, e^{i\omega t}\, d\omega$

Questions? Assumptions above?

... so let's rejoin the real world ....

real data & discrete (meaning?)

on board, then

Lecture_8_etc A + AA

data ^(usually) sampled at some simple freq ν_s that we get to choose

Rule 1 — Nyquist?

once ν_s set, can't see" freq $> \frac{\nu_s}{2} = \nu_{Nyquist}$

consider ML plot, $\nu_s = \frac{1}{hr}$ and have $N = 200$ pts

what is the largest period (smallest freq) we can hope to see" (resolve)

for WES

L7.30

$T_{samp} = N\Delta t = \frac{N}{\nu_s} = 200\ hrs$

$\Rightarrow \nu_e = \frac{1}{200\,hrs}$ in this case

So choices of ν_s + N set highest + lowest resolvable ν_s

$$\nu_e < \nu < \nu_n$$

$\nu_n = \frac{\nu_s}{N}$ ← $\frac{3\nu_s}{N}$

$\nu_n \geq \frac{\nu_s}{2}$ → $\frac{\nu_s}{4}$

go here for next time

practical

are we done? No there's "aliasing" + "folding" but first, more Fourier theory:

Thinking about our integral and discrete data we are really posting that

theory $x(t) = A_0 + A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(2\omega_1 t + \phi_2) + \dots A_n \cos(n\omega_1 t + \phi_n)$

Q: do we know $x(t)$ for all time? rarely!

so what is $\omega_1 = \frac{2\pi}{\tau}$ where $\tau = N\cdot\Delta t = \frac{N}{\nu_s}$

Do Fitted Fourier Approx here

(L 7.3)

So $\omega_f = \frac{2\pi f_s}{N}$

Note that series is constructed of cos's with integer multiples of ω_f and phases

$1\omega_f, 2\omega_f, \ldots N\omega_f$

but wait $N\omega_f = 2\pi f_s$, a beg? Nyquist?

ok so $y(t) = A_0 + A_1 \cos(\omega_f t + \phi_1) + \ldots$
$A_{\frac{N}{2}} \cos(\frac{N}{2}\omega_f t + \phi_{\frac{N}{2}})$

We can alternately write this as:

$$y(t) = C_0 + \sum_{n=1}^{N/2} \left[C_n e^{in\omega_f t} + C_n^* e^{-in\omega_f t} \right]$$

C_0, C_n's called "Fourier coefficients"

F.Cs go each with their own freq ($n\omega_f$)

- FCs are complex, real + imag part (or amp + phase)

This works b/c of something called "orthogonality"

The cosines or $e^{i\omega t}$'s are called an orthonormal basis set and they exhibit orthogonality:

$$\sum_{\text{cycle}} \cos(\omega_a t + \phi_a) \cdot (\cos \omega_b t + \phi_b) =$$

window

$\emptyset$ if $\omega_a \neq \omega_b$

constant if $\omega_a = \omega_b$

theory

↑

ML Examples B + C

↓

practical

back to the practical — aliasing & fold-over

aliasing means signals above Nyquist ($\frac{\nu_s}{2}$) don't just go away, they "fold over" and mimic legitimate data. This is worrisome!

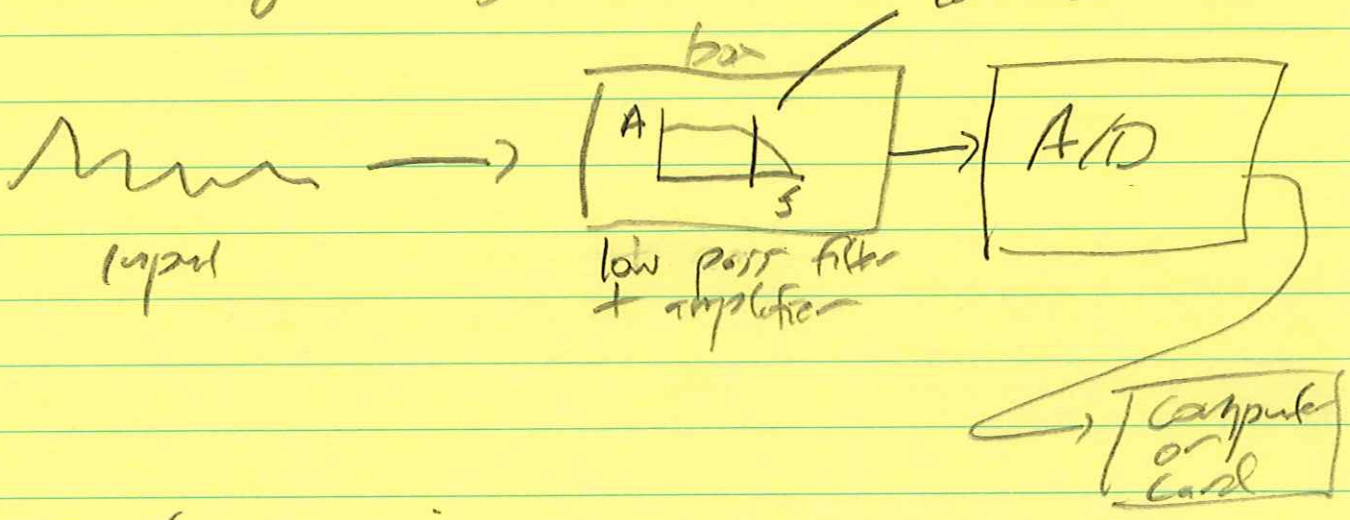
ML Examples D + E

talk about bars in Fig 2 plot, what do they mean?

Phys 311
~~Sp 14~~
Lec 7

~~Ex 4~~
~~ω15~~
~~7.8~~

So how do we design our (discrete) data aquisition systems?



So, to summarize

data is always discrete time series, defined by sample freq ν_s + N # of points

aliasing can occur for signal $\nu > \nu_N$

typically think about freq range of interest (and amplitude), design LP filter/amp + A/D to collect data

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3B FFT background

postulate that $y(t) = A_0 + A_1 \cos(\omega_1 t + \phi_1) + A_2 \cos(2\omega_1 t + \phi_2) + \ldots A_n \cos(n\omega_1 t + \phi_n)$

Go to page 2 of Lec 7 notes

A AA
B C
D E

Phys 391

~~Fig~~ WIS

~~L8.1~~

L7.6

1) How orthogonality works

2) Fourier analysis / reconstruction
2a) Fourier synthesis

3) Fourier properties

$$\psi(x_i + k\Delta x)\,[\text{or } \psi(x,y,z)] = C_0 + \sum_{n=1}^{N-1}\left[C_n e^{i n \Delta\omega_s k\Delta t} + C_n^* e^{-i n \Delta\omega_s k\Delta t}\right]$$

- with $i = \sqrt{-1}$, $\omega_s = \frac{2\pi\nu_s}{N}$, Δt (or $\Delta x, \Delta y, \Delta z$) is discrete ⇐
- n ranges from 1 to N
- C_0, C_n's are Fourier Coeffs.
- $C_n^* = \text{real}(C_n) - i \cdot \text{imag}(C_n)$

once C_n's are determined (analysis of spectrum) this formula would give us our reconstruction of $\psi(x_i + k\Delta x)$

1) We use orthogonality to Fourier analyze our data, find C_0 & C_n's (latter are complex)

(show how phase works)

~~Lec~~ wk5

~~L8.2~~

L7.1

since, by orthogonality, a particular basis frequency fctn (essentially a cosine wave) will "pick out" like frequencies in $y(t)$ we do this

$$G(w_n) = \Delta t \sum_{k=0}^{N-1} y(t_k)\, e^{-i2\pi \frac{nk}{N}}$$

($y(t_k)$: all data; $e^{-i2\pi \frac{nk}{N}}$: basis fctn)

↳ summing over time steps

this gives us the nth G_n, so we need to do this for all our basis fctns, e^{-iwt}

where $w_n = \frac{2\pi \delta_s}{N} \cdot n \quad \left(n = -\frac{N}{2}+1, \ldots, 0, \ldots \frac{N}{2}\right)$

note $G(w_n) = \Delta t\, c_n$ ← what we're talking about

also G's, c_n's are complex, have amp & phase

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2) Lec 8 - etc F) + G)

finally, we can turn this around, if we know c_n's, then

$$y(t_k = k\Delta t) = \sum_{n=0}^{N-1} G(w_n)\, e^{i2\pi \frac{nk}{N}}$$

inverse FT can ⇒ individual pts of time series by summing over c_n's